# Interference, reduced action, and trajectories

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**Abstract**

Instead of investigating the interference between two stationary, rectilinear wave functions in a trajectory representation by examining the trajectories of the two rectilinear wave functions individually, we examine a dichromatic wave function that is synthesized from the two interfering wave functions. The physics of interference is contained in the reduced action for the dichromatic wave function. As this reduced action is a generator of the motion for the dichromatic wave function, it determines the dichromatic wave function's trajectory. The quantum effective mass renders insight into the behavior of the trajectory. The trajectory in turn renders insight into quantum nonlocality.

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## 1 INTRODUCTION

The trajectory representation of quantum mechanics is a nonlocal, phenomenological theory that is deterministic. The quantum Hamilton-Jacobi equation underlies the trajectory representation of quantum mechanics.[1,2] The underlying Hamilton-Jacobi formulation couches the trajectory representation of quantum mechanics in a configuration space, time domain rather than a Hilbert space of wave mechanics. Faraggi and Matone, using a quantum equivalence principle that connects all physical systems by a coordinate transformation, have independently derived the quantum stationary Hamilton-Jacobi equation (QSHJE) without using any axiomatic interpretations of the the wave function, $\psi$.[3,4] With Bertoldi, they have extended their work to higher dimensions and to relativistic quantum mechanics.[5] Without axiomatic interpretations, the trajectory representation has been used to investigate the foundations of quantum mechanics free of Copenhagen philosophy. The trajectory representation has microstates that provides a counterexample showing that $\psi$ is not an exhaustive description of quantum mechanics.[1,4,6–9] The trajectory interpretation has shown that the initial values of position and momentum, of the Heisenberg uncertainty principle, form an insufficient subset of initial values for determining the particular complete solution of the QSHJE: acceleration and jerk, or their equivalents, must be included to form a necessary and sufficient set of initial values to determine a unique trajectory.[2,4,6,10] The trajectory representation also describes quantization [1,4,7,8] and tunnelling without probability.[4,11] Other aspects of the trajectory representation have been studied.[12,13]

Interference is another fundamental attribute of quantum mechanics that distinguishes it from classical mechanics. Interference implies quantum entanglement, which in turn implies nonlocality. An initial assessment of interference by the trajectory representation was noted as it incidently arose in the study of transmission and reflection by a semi-infinite step barrier.[14] We examine interference in detail to gain new

insight. In the Copenhagen interpretation, the degree of interference between two plane wave functions, $\psi_1(x)$ and $\psi_2(x)$ is manifested by the variation of Born's probability density for the sum of entangled wave functions as a function of position $x$, that is

$$\big|[\psi_1(x)+\psi_2(x)]^\dagger[\psi_1(x)+\psi_2(x)]\big|.$$

We examine herein the manifestation of interference in the deterministic trajectory interpretation of quantum mechanics. The approach that we use considers a solitary dichromatic particle whose spectral components interfere with each other rather than studying the interference between one aggregate of particles and another. In practice, we spectrally sum $\psi_1(x)$ and $\psi_2(x)$, each a monochromatic wave function representing a spectral component, to form a dichromatic wave function, $\psi_d(x)$, for a dichromatic particle. From $\psi_d(x)$, we can determine the dichromatic reduced action (Hamilton's characteristic function) and trajectory. We find that trajectory representation describes destructive interference and reinforcement as a dwell-time phenomenon of the trajectory weighted by Faraggi and Matone's quantum equivalent mass,(4) so invoking probability and Born's probability density is unnecessary. We additionally find that the trajectory representation describes phenomenon that has been attributed to quantum-mechanical spreading of the wave function by the Copenhagen interpretation. We also find that the consequent trajectories are nonlocal.

We investigate herein the prototype of interference problems, interference between two stationary plane wave functions with the same wavelength. This problem has been examined by Holland using Bohmian mechanics.(15) The findings of the trajectory representation differ with those of Bohmian mechanics as the two representations have different equations of motion even though both representations derive the same reduced action from the same QSHJE.(2,16)

As part of this investigation, we show that while the trajectory representation has analogies to group velocity of wave packets, the trajectory representation is more general.

In a companion article, *welcher Weg* is examined for a simplified quantum Young's diffraction experiment.(17)

In Sect. 2, we investigate interference in one dimension in a trajectory representation. The interfering wave functions are used to synthesize a dichromatic wave function. The dichromatic reduced action is developed as the generator of the motion of interference patterns. Dwell times are developed for interference. Nonlocal propagation is exhibited. In Sect. 3, we develop the contours of constant reduced action in two dimensions. The evolution of the reduced action as it goes from representing a running wave function through a dichromatic wave function to a standing wave function is developed. Interference is shown to cause the trajectory not to be orthogonal to the contours of reduced action in general. In Sect. 4, the conclusions are summarized.

## 2 EQUATION OF MOTION

Let us first examine the simple case of interference between two rectilinear, monochromatic wave functions of equal wavelength in one dimension. One wave function, $\psi_+(x) = A\exp(ikx)$, propagates in the positive $x$-direction; the other,$\psi_-(x) = B\exp(-ikx - i\beta)$, in the negative $x$ direction. The amplitudes $A$ and $B$ are real, non-negative constants, $+k$ and $-k$ are respectively the wave numbers for the interfering wave functions $\psi_+$ and $\psi_-$, and $\beta$ is a constant phase shift in $\psi_-$. Hence we only need one dimension to investigate this interference. We arbitrarily assume that $A > 0$ and $A \geq B \geq 0$. Alternatively, this interference may be considered to be the interference between a running wave function, $(A - B)\exp(ikx)$, and a standing wave function, $2B\exp(-i\beta/2)\cos(kr + \beta/2)$.

We may spectrally synthesize a dichromatic wave function, $\psi_d(x)$, with only two spectral components, $+k$ and $-k$. The subscript "$d$" in any function, for example $f_d(x)$, will designate that this function describes some property of the dichromatic wave function. The dichromatic wave function is a time-independent solution of the Schrödinger equation by the superpositional principle of linear homogeneous differential equations. As such, $\psi_d(x)$ is the wave function for the dichromatic particle. Historically in quantum mechanics, the dichromatic wave function was first developed by a trajectory representation in Ref. 18 (where it was identified as a compoundly modulated wave) rather then by a Schrödinger representation. The complete solution

for the reduced action of the QSHJE rendered compoundly modulated (dichromatic) wave functions.[18] The dichromatic wave function is described by[15,18]

$$\begin{aligned}\psi_d(x) &= \psi_+(x) + \psi_-(x) \\ &= [A^2 + B^2 + 2AB\cos(2kx+\beta)]^{1/2} \exp\left[i \arctan\left(\frac{A\sin(kx) - B\sin(kx+\beta)}{A\cos(kx) + B\cos(kx+\beta)}\right)\right]. \end{aligned} \tag{1}$$

The two running wave functions, $\psi_+$ and $\psi_-$, are now spectral components, $+k$ and $-k$, of $\psi_d$. The dichromatic wave function, $\psi_d$, innately incorporates the interference between its two spectral components. We consider that our particle of interest is a dichromatic particle whose wave function is $\psi_d(x)$. In the limit $A \to B$, then the last line of Eq. (1) still renders $\lim_{A\to B}\psi_d(x) = 2B\cos(kx), i2B\sin(kx)$ for $\beta = 0, \pi$ respectively. Destructive and constructive interference is manifested by the cosine term in the amplitude of the dichromatic wave function $\psi_d$ in Eq. (1). The Copenhagen interpretation of quantum mechanics attributes Born's probability density to the absolute value of $\psi_d^\dagger(x)\psi_d(x)$. Note that $\psi_+$, $\psi_-$ and $\psi_d$ are all solutions of the time-independent Schrödinger equation for the free particle,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = \frac{\hbar^2 k^2}{2m}\psi$$

where $\hbar$ is Planck's constant and $m$ is particle mass. The energy, $E$, for the free dichromatic particle is specified by $E = \hbar^2 k^2/(2m)$.

We recognize that there is a degree of arbitrariness in specifying that the dichromatic wave function, $\psi_d$ is synthesized by the interference of two running wave functions, $\psi_+$ and $\psi_-$. As another dichromatic wave function,

$$\psi_{dd} = [A^2 + B^2 - 2AB\cos(2kx+\beta)]^{1/2} \exp\left[i \arctan\left(\frac{A\sin(kx) + B\sin(kx+\beta)}{A\cos(kx) - B\cos(kx+\beta)}\right)\right],$$

is also a solution of the Schrödinger equation and is, for finite $A/B$, independent of $\psi_d$, then $\psi_+ = \psi_d + \psi_{dd}$. By the superpositional principle of linear homogeneous differential equations, one can always specify that the running wave function $\psi_+$ is formed by the interference between two dichromatic wave functions, $\psi_d$ and $\psi_{dd}$. All sets of independent solutions of the time-independent Schrödinger equation are equally valid. For this investigation, we chose one solution to be $\psi_d(x)$ for heuristic purposes.

The reduced action $W_d$ for the dichromatic wave function is given by

$$W_d = \hbar \arctan\left(\frac{A\sin(kx) - B\sin(kx+\beta)}{A\cos(kx) + B\cos(kx+\beta)}\right). \tag{2}$$

Likewise, had we been interested in free particles with one spectral component, then the reduced actions for $\psi_+$ would be given by $W_+ = +\hbar kx$; for $\psi_-$, by $W_- = -\hbar kx$. Note that $W_d$, $W_+$ and $W_-$ are all solutions of the QSHJE for the free particle

$$\frac{(\partial W/\partial x)^2}{2m} - \frac{\hbar^2 k^2}{2m} + \underbrace{\frac{\hbar^2}{4m}\left[\frac{\partial^3 W/\partial x^3}{\partial W/\partial x} - \frac{3}{2}\left(\frac{\partial^2 W/\partial x^2}{\partial W/\partial x}\right)^2\right]}_{\text{Bohm's quantum potential, } Q} = 0. \tag{3}$$

We shall demonstrate the trajectory representation of interference by investigating the reduced action, $W_d$, for the dichromatic wave function as exhibited by Eq. (2). Interference between the two spectral components of $\psi_d$ is innate in $W_d$. The reduced action, $W_d$, is the generator of the motion for manifesting interference and the dynamics of the dichromatic wave function.

The conjugate momentum, $p_d$, for the dichromatic wave function is defined as the gradient of the reduced action. For the dichromatic wave function in one dimension, the conjugate momentum is

$$p_d \equiv \frac{\partial W_d}{\partial x} = \frac{\hbar k(A^2 - B^2)}{A^2 + B^2 + 2AB\cos(2kx + \beta)}. \tag{4}$$

The interference is manifested in the conjugate momentum by the cosine term in the denominator of Eq. (4). The conjugate momentum of the dichromatic wave function is by Eq. (1) inversely proportional to the inverse square of amplitude of the dichromatic wave function. Hence, Born's probability density, $\rho_d(x)$ is related to the conjugate momentum by Eqs. (1) and (4) as

$$\rho_d(x) = \psi_d^\dagger(x)\psi_d(x) \prec [(\partial W_d/\partial x')^{-1}]|_{x'=x}, \tag{5}$$

for the dichromatic wave function. Thus, the conjugate momentum for the dichromatic wave function provides an alternative explanation of interfering wave functions without any need to invoke Born's probability density. For completeness, we note that for bound states where $\psi$ is real, then Eq. (5) does not hold.[9] This will be discussed further later.

We use Jacobi's theorem to determine the quantum equation of motion.[1,2,4,6] As Jacobi's theorem also determines the equation of motion in classical mechanics, Jacobi's theorem is universal transcending across the division between classical and quantum mechanics. In the classical limit, trajectories are consistently determined. Note that this is a departure from Bohmian mechanics, which would have assumed that $p_d$ be the mechanical momentum $m\dot{x}_d$, from which the Bohmian quantum equation of motion be determined by integrating $p_d$.[4,8] For comparison, the Bohmian trajectory for interference between plane wave functions has been given by Holland.[15] Jacobi's theorem renders

$$t_d - \tau = \partial W_d/\partial E = \frac{mx(A^2 - B^2)}{\hbar k[A^2 + B^2 + 2AB\cos(2kx + \beta)]} \tag{6}$$

where $\tau$ is the Hamilton-Jacobi constant coordinate (a nontrivial constant of integration) in units of time necessary for determining the motion of the dichromatic wave function.

Faraggi and Matone have shown that $\dot{x}_d$ may be expressed from Eq. (6) by[4]

$$\dot{x}_d = \frac{1}{\partial t_d/\partial x} = \frac{1}{\partial^2 W_d/\partial x \partial E} = \frac{1}{\partial^2 W_d/\partial E \partial x} = \frac{1}{\partial p_d/\partial E}. \tag{7}$$

We designate $\partial t_d/\partial x$ to be dwell density where $\int_{x_1}^{x_2} \partial t_d/\partial x \, dx = t_d(x_2) - t_d(x_1)$ renders the dwell time that the dichromatic particle spends in the domain between $x_1$ and $x_2$. Before we evaluate the velocity, we present a brief digression. Let us now associate the constant $E$ with some time-independent quantum Hamiltonian $\mathcal{H}(x,p)$ such that $\mathcal{H} = E$ while $\mathcal{H}$ still obeying the Faraggi-Matone quantum equivalence principle (the subscript $d$ is dropped in this digression). A velocity, analogous to Eq. (7), may then be expressed as

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p}$$

which is half of the canonical equations of Hamilton for time independence where it is noted that $p$ is the conjugate momentum and not the mechanical momentum. The other half follows from $\mathcal{H}(x,p)$ being a constant and is given by

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial x}.$$

This digression has been presented for heuristic purposes only. It shows that quantum Hamilton-Jacobi representation would be consistent with the the canonical equations for a quantum $\mathcal{H}(x,p)$ analogous to classical mechanics. Regrettably, we presently do not know the algorithm to develop $\mathcal{H}(x,p)$. The problem is that a reversible canonical transformation is needed to generate the classical Hamilton's principal function from the classical Hamiltonian, $H(x,p_{\mathrm{mech}})$ where $p_{\mathrm{mech}}$ is the mechanical momentum, $p_{\mathrm{mech}} = m\dot{x}$. On the other hand, the Faraggi-Matone quantum equivalence principle is restricted to just coordinate transformations.[4]

The interested reader who desires to know more on applying trajectory representation of quantum mechanics in a non-Hamilton-Jacobi formulation is referred to Bouda et al[13] for progress in "quantum analytical mechanics", to Wyatt[16] for progress in "quantum trajectory methods", and to Poirier[19] for progress in "counterpropagating wave method".

We now return to Eq. (7)and evaluate the velocity, $\dot{x}_d$, for the dichromatic wave function as

$$\dot{x}_d = (\partial t/\partial x_d)^{-1} = \frac{\hbar k}{m(A^2-B^2)} \frac{[A^2+B^2+2AB\cos(2kx+\beta)]^2}{A^2+B^2+2AB\cos(2kx+\beta)+4ABkx\sin(2kx+\beta)}. \tag{8}$$

From Eqs. (4) and (8), the conjugate momentum for the dichromatic wave function is not the mechanical momentum, i.e., $p_d \neq m\dot{x}_d$.

Faraggi and Matone introduced an "effective quantum mass", $m_Q$, such that $m_Q\dot{x} = \partial W/\partial x$ or $m_Q = m(1-\partial Q/\partial E)$ where $Q$ is Bohm's quantum potential in Eq. (3).[4] Faraggi and Matone were inspired by special relativity to introduce $m_Q$. Solid state physics provides a precedence where the effective mass, $m^*$, for electrons and holes accounts for the interaction between an electron or hole and a crystal as the electron or hole responds to an external force. In the band theory of solids, $m^*$ may be positive, negative, zero or infinite.

For the dichromatic wave function, the effective quantum mass, $m_{Q_d}$ is given by

$$m_{Q_d} = m\frac{(A^2-B^2)^2[A^2+B^2+2AB\cos(2kx+\beta)+4ABkx\sin(2kx+\beta)]}{[A^2+B^2+2AB\cos(2kx+\beta)]^3}. \tag{9}$$

For the dichromatic wave function by Eq. (5), the Born's probability density, $\rho_d$ is proportional to dwell density $\partial t_d/\partial x$ divided by $m_{Q_d}$ as

$$\rho_d = \psi_d^\dagger(x)\psi_d(x) \prec \frac{1}{\partial W_d/\partial x} = \frac{1}{m_{Q_d}\dot{x}} \prec \partial t/\partial x. \tag{10}$$

Thus, the trajectory representation of interference need not attribute a probability amplitude to the dichromatic wave function $\psi_d(x)$.

For completeness, we note that if the wave function is real, then $\psi^\dagger(x)\psi(x) \neq (\partial W/\partial x)^{-1}$. Elsewhere, the trajectory interpretation has used the concept of dwell time to describe phenomena for real $\psi(x)$. The trajectory representation predicts the dwell times for tunnelling[11] and reflection[20] that are consistent with those of Barton[21], Hartman[22], and Fletcher.[23] Also, the trajectories of bound state square wells at their turning points at $x = \pm\infty$ manifest that $\dot{x} \to \pm\infty$ while $\partial W/\partial x \to 0$ implying that the trajectory deep in the forbidden region transits an infinite distance in a finite time.[20] A suggested generalization for the effective quantum mass is

$$m_Q = \frac{\partial t/\partial x}{\psi^\dagger(x)\psi(x)}$$

for real or complex $\psi(x)$.

Let us now investigate a particular case. We consider the case that $\hbar = 1,\ m = 1,\ k = \pi/2, A = 1,\ B = 0.5,\ \beta = 0, \pi$ and $\tau = 0$. The trajectories for $\beta = 0, \pi$ are exhibited on Fig. 1 where the trajectories for the dichromatic wave function exhibit quasi-periodic reversals in time where $\partial t_d/\partial x = 0$. These time reversals occur in the vicinity of $2kx + \beta = n\pi,\ n = 1, 2, \cdots$. The time reversals induce nonlocality on two counts. First, the time reversals as extremum in time induce an instantaneous infinite velocities in accordance with Eq. (7). Second, the time reversals imply pair creations and annihilations of interference patterns. For $B/A \ll 1$, latent early time reversals may be suppressed. The upper time reversals manifest destructive interference where the absolute value $\psi_d$ form a local minima. Likewise, the lower time reversal manifest constructive interference. By Eq. (6) for $A = 1$ and $B = 0.5$, the trajectories in Fig. 1 have the locus of their upper time reversals, $t_u$ on a straight line in the $t, x$-plane given by

$$t_u = \frac{A+B}{A-B}\frac{mx}{\hbar k} \tag{11}$$

for any phase shift $\beta$. Likewise, the locus of lower time reversals, $t_\ell$, that manifest constructive interference (reinforcement) are located along the straight line in the $t, x$-plane given by

$$t_\ell = \frac{A-B}{A+B}\frac{mx}{\hbar k} \tag{12}$$

for any phase shift $\beta$. The lines $t_u(x)$ and $t_\ell(x)$ form wedge that is densely covered by trajectories launched from origin of Fig. 1 by adjusting the phase factor $\beta$. As the trajectories are horizontal (parallel to the $t$-axis) at these reversal points, the lines $t_u$ and $t_\ell$ are not caustics of the trajectories for finite $A$ and $B$ and $A \neq B$. As illustrated by Fig. 2, additional trajectories for $\beta = \pi/4, \pi/2, \cdots, 7\pi/4$ show the forming of caustics near the lines of time reversal but outside the wedge formed by $t_u$ and $t_\ell$. The area in the $t, x$-plane between the caustics is designated the extended wedge. The time reversals of the trajectories ensure that a solitary trajectory continues to span the extended wedge quasi-periodically.

While the trajectories for the dichromatic wave function have time reversals, this does not imply that the dichromatic wave function itself reverses direction. Readers interested in the phenomenon of direction reversing travelling (running) waves are referred to Knobloch et al.(24)

In order to gain insight, we now suspend our investigation of the time reversal phenomenon exhibited by Fig. 1 until after we consider two limiting cases: first, the running wave function only case (i.e., $B = 0$); and, second, the standing wave function only case (i.e., A=B=1 for a cosine wave function). For the first limiting case, if we let $B \to 0$ for our particular case, then $\psi_d = A\exp(ikx)$ as the dichromatic wave function becomes a pure running wave function only. As $B \to 0$, then $t_u \to t_\ell$ by Eqs. (11) and (12). The equation of motion, Eq. (6) becomes $t_d = mx/(\hbar k)$ eliminating all time reversals. The velocity becomes $\dot{x}_d = \hbar x/m$, and the effective quantum mass becomes $m_{Q_d} = m$ as expected.

For the second limiting case, we let $A = 1 = B + \epsilon$ where $\epsilon$ is real and nonnegative so that as $\epsilon \to 0$, then $B$ approaches $A$ from below. For $\lim_{\epsilon\to 0}\psi_d = 2A\cos(kx)$ and the two edges of the wedge given by Eqs. (11) and (12) become orthogonal to each other so that the wedge spans the entire quadrant $t, x \geq 0$ of $t, x$-plane in Fig. 1 as the dichromatic wave function becomes entirely a standing wave function. The equation of motion, Eq. (6), for this standing wave function would then become

$$\lim_{\epsilon\to 0} t_d = \sum_{n=1}^{\infty}\{\delta[x-(2n-1)\pi/(2k)] + \delta[x+(2n-1)\pi/2k]\} \tag{13}$$

where $\delta$ is the Dirac delta function. Equation (13) presents the motion for a standing wave function whose launch point is at $x = -\infty$ with the constant coordinate $\tau = 0$. But the standing wave function whose launch point is at $x = 0$ is the more interesting case to examine. Thus, we shall restrict Eq. (13) to the domain $x > 0$ where Eq. (13) becomes

$$\lim_{\epsilon\to 0} t_d = \sum_{n=1}^{\infty}\delta[x-(2n-1)\pi/(2k)] = \sum_{n=1}^{\infty}\{\delta[x-(2n-1)], \quad x > 0. \tag{14}$$

In the domain $x < 0$, we now relax the requirement that $B < A$. We use the subscript $-d$ to denote a property of the dichromatic wave function for with $B > A$. In the alternative representation, $\psi_{-d}$ represents the interference between a running wave function in the negative $x$-direction, $(B-A)\exp(-ikx)$ and a standing wave function $2A\cos(-kx)$. The corresponding reduced action is still $W_{-d} = \hbar\arctan\left(\frac{A-B}{A+B}\tan(kx)\right)$ consistent with Eq. (2). The equation of motion is still rendered by Eq. (6), but since $B > A$, motion is in the negative $x$-direction. We let $A = B - \epsilon$ where $\epsilon$ is real and nonnegative so that as $\epsilon \to 0$, then $B$ approaches $A$ from above. The equation of motion, Eq. (6), for this standing motion in the domain $x < 0$ is

$$\lim_{\epsilon \to 0} t_{-d} = -\sum_{n=1}^{\infty} \delta[x-(2n-1)\pi/(2k)] = -\sum_{n=1}^{\infty} \{\delta[x-(2n-1)], \quad x<0. \tag{15}$$

Thus, the standing wave function with launch point at $x=0$ has positive infinite velocity for $x>0$ and negative infinite velocity of propagation for $x<0$ except at the nulls, $x=\pm1,\pm2,\pm3,\cdots$, of the the standing wave function, $2\cos(kx)$, where the null innately has nil velocity of propagation. This is consistent with the concept of a standing wave function. The effective quantum mass would become

$$\begin{aligned} \lim_{\epsilon \to 0} m_{Q_{\pm d}} &= 0, \quad x \neq \pm1,\pm3,\pm5,\cdots \\ &= \infty, \quad x=\pm1,\pm3,\pm5,\cdots. \end{aligned}$$

We note that the effective quantum mass becomes infinite where the velocity of the standing wave function is nil and becomes nil where the velocity is infinite.

We now return to investigating the particular case exhibited in Fig. 1 where $A=1$, $B=0.5$, $k=\pi/2$ and $\beta=0$. The motion of propagation of the dichromatic wave function, $\psi_d(x)$ is a combination of the running wave function $(A-B)\exp(ikx)=2^{-1}\exp(i\pi x/2)$ and the standing wave function $2B\cos(kx)=\cos(\pi x/2)$. As $B$ becomes smaller for $B/A\leq1$, the wedge between the upper and lower turning points, as given by Eqs. (11) and (12), narrows as the motion becomes more like a running wave function. Conversely, as $B$ increases for $B/A\leq1$, the allowed wedge of propagation widens manifesting the increasing influence of the standing wave function.

Another way of examining the propagation implied by Fig. 1 uses the effective quantum mass, $m_Q$, of Faraggi and Matone, Eq. (9) and the velocity, $\dot{x}_d$ of Eq. (8). Both $m_{Q_d}$ and $\dot{x}_d$ are poorly behaved for our particular case, so we tame their behavior with the Euler-inspired transforms

$$\mathcal{M}_{Q_d} = 0.8\ \mathrm{sgn}(m_{Q_d})\frac{m_{Q_d}^2}{1+m_{Q_d}^2} \quad \text{and} \quad \dot{\mathcal{X}}_d = \mathrm{sgn}(\dot{x}_d)\frac{\dot{x}_d^2}{1+\dot{x}_d^2}.$$

We plot the transformed effective quantum mass, $\mathcal{M}_{Q_d}$, and the transformed velocity, $\dot{\mathcal{X}}_d$, on Fig. 3 as function of $x$. For clarity of exposition, $\mathcal{M}_{Q_d}$ is normalized differently than $\dot{\mathcal{X}}_d$. As exhibited by Fig. 3, the effective quantum mass, $m_{Q_d}$ for the dichromatic wave function offers insight into the nonlocality at the time reversal points. The effective mass, $m_{Q_d}$ is negative wherever the trajectory exhibits retrograde motion with respect to time. The effective quantum mass has the property that $m_{Q_d}=0$ for the dichromatic wave function wherever $\dot{x}_d\to\pm\infty$ at the extrema in time denoting the time reversal points where trajectory segments with $m_{Q_d}$ of opposite sign are either created or annihilated. The local extrema in time at the turning points induce an instantaneous infinite velocity. Still, the nil quantum effective mass for the dichromatic wave function at the turning points implies that pair creation would not be an endoergic process and that pair annihilation likewise would not be an exoergic process. Hence, neither creation nor annihilation of pairs of interference patterns implies a high energy process, which is consistent with $\psi_d$ and $p_d$ being solutions for energy $E=\hbar^2k^2/(2m)$ to the Schrödinger equation and QSHJE respectively. For completeness, Fig. 3 is only quasi-periodic.

As exhibited by Fig. 1, the trajectory segments are annihilated near $x=1,3,5,\cdots$ where by Eq. (2), the reduced action for the particular case becomes $W_d=\arctan[3^{-1}\tan(x\pi/2)]$ inducing jumps to new Riemann sheets at $x=1,3,5,\cdots$. The annihilations just manifest the confluence of two interference patterns that each have reached their point of maximum destructive interference. Likewise, the creation of two trajectory segments near $x=2,4,6,\cdots$ manifests the launching of two interference patterns travelling in opposite directions — one pattern in the advancing direction while the other in retrograde with respect to time. As the interference patterns travel from their lower turning points to the upper turning points, the interference goes monotonically from constructive to destructive.

The study of the group velocity[25] of wave packets in a general wave propagation renders a precedent. The group velocity is given by the partial derivative of frequency with respect to wavelength is reminiscent of Jacobi's theorem. Group velocity is only a valid concept as long as the wave packet maintains its integrity. Any creation or annihilation of a pair of interference patterns are sufficient to disrupt the integrity of the wave packet. In the general theory of wave packet propagation, integrity is maintained for a finite duration by having a carrier frequency that is very large in comparison to the frequency modulation. The trajectory representation renders a generalization of the group velocity on two counts. First, as the dichromatic wave function, $\psi_d$, given by Eq. (1) has no carrier wave, the trajectory representation for the dichromatic wave function has generalized the group velocity concept to cover situations without a carrier wave. Second, the trajectory representation relaxes the requirement that the dichromatic wave function have spectral components with approximately equal strength (this relaxation can be extended to polychromatic wave functions). As the integrity of interference patterns for the individual trajectory segments for dichromatic wave function is comparatively short in the trajectory representation, the transition to other trajectory segments, whose motion is in the opposite direction, is accomplished by pair creations and annihilations of interference patterns at the turning points where the quantum effective mass goes to zero. The trajectory representation thus extends trajectories beyond the point where the wave packet loses its integrity.

There is an alternative explanation to the loss of integrity of the wave packet and the generation of pair creations and annihilations of interference patterns. Any finite self-entanglement innately prescribes that as the dichromatic wave function propagates away from it launch point at $x_0$ along its trajectory, it eventually reaches a point where the interfering components, $\psi_1 = A\exp[ik(x-x_0)]$ and $\psi_2 = B\exp[-ik(x+x_0)]$, have separated sufficiently far, $2x$, to induce the loss of wave packet integrity. There, self-entanglement compels the nonlocal generation of pair creation and annihilations of interference patterns. In this manner, self-entanglement, for which nonlocality is inherent as $\psi_d \neq K\psi_1\psi_2$ where $K$ is some constant, manifests strong nonlocality for the dichromatic particle can be at several separated locations simultaneously. The underlying physics for this strong nonlocality is that the dichromatic particle travels between particular points $x_1$ and $x_2$ in one-dimensional space along a series of trajectory segments that alternate between forward and retrograde motion with respect to time such that the transit time between $x_1$ and $x_2$ is nil. As the dichromatic particle may be simultaneously in multiple locations, the nonlocal dichromatic particle is a distributed particle, and its trajectory manifests motion for a distributed particle.

For completeness, we present a Copenhagen interpretation of quantum mechanics of this Section. Copenhagen rejects determinism and trajectories out of hand but does accept the concept of wave packets. By Copenhagen, the wedge of allowed trajectories in Fig. 1 would manifest that, first, the wave packet of the dichromatic wave function was launched at the apex of the extended wedge at $x = 0$ and $t = 0$. Second, by Copenhagen, as this wave packet subsequently propagates away from the apex of the extended wedge, the wave packet would spread with time as a function of the broadness of the wedge. Third, this wave packet by Copenhagen would eventually spread so far that it would loose its integrity where the trajectory representation manifests a time reversal (retrograde motion). However, by Copenhagen, loss of wave packet integrity would not preclude further spreading. Copenhagen would also attribute the measurement of $x$ at time $t$ to a collapse of the dichromatic wave function to the position $x$ whose distribution of measurement positions would manifest only a Born probability density, $\psi_d^\dagger(x)\psi_d(x)$, and not some underlying trajectory of the dichromatic wave function. Finally, Copenhagen would agree that the dichromatic wave function, $\psi_d(x)$, would be an entangled combination of $\psi_1(x)$ and $\psi_2(x)$ as $\psi_d(x)$ has already been shown to be not factorable into $\psi_1(x)$ and $\psi_2(x)$.

# 3 REDUCED ACTION AND TRAJECTORY EQUATION

In this Section, we exhibit in two dimensions the contours of reduced action that represent interference and develop the trajectory equation. Let us consider the reduced action for a wave function given by

$$\psi(x,y) \quad = \quad [A\exp(ik_x x) + B\exp(-ik_x x)]\exp(ik_y y)$$

$$= \quad [A^2 + B^2 + 2AB\cos(2k_x x)]^{1/2} \exp\left[i \arctan\left(\frac{A-B}{A+B}\tan(k_x x)\right) + ik_y y\right], \tag{16}$$

which is separable. The wave function¡ $\psi(x,y)$, is a solution of the Schrödinger equation for the free particle of energy $E = \hbar^2(k_x^2 + k_y^2)/(2m)$. The right side of the first line of Eq. (16) represents two wave functions, each also a solution to the same Shrödinger equation, that interfere with each other. The right side of the second line of Eq. (16) is a wave function and with innate internal interference and has been synthesized from the two wave functions of the right side of the first line of Eq. (16). This synthesized wave function is denoted herein as $\psi_i$ (dichromatic wave is no longer proper terminology for the spectral component $k_y$ adds a third spectral component). The reduced action, $W_i$, for $\psi_i$ is given by

$$W_i = \hbar\left[\arctan\left(\frac{A-B}{A+B}\tan(k_x x)\right) + k_y y\right] \tag{17}$$

where $y$ is a cyclic coordinate. The trajectory equation which renders the constant coordinate $y_0$ in accordance with Jacobi's theorem is given by

$$y_0 = \frac{\partial W}{\partial k_y} = y - \frac{(A^2-B^2)(k_y x)/k_x}{A^2+B^2+2AB\cos(2k_x x)}. \tag{18}$$

Let us consider a particular case. We let $A = 1$, $B = 0.5$, $\hbar = 1$, $m = 1$ and $k_x = k_y = \pi/2$. We present selected contours of constant reduced action, $W$ of Eq. (17), as solid lines in Fig. 4. These selected contours are separated by $h/4$ units of action (we have set $\hbar = 1$). Due to the periodic $x$-dependence on the right side of Eq. (17), the contours of constant reduced action diagonally serpentine across the infinite $x, y$-plane. We also present on Fig. 4 the trajectory whose launch point is the origin ($y_0 = 0$) as a dashed line by applying Eq. (18) to this particular case.

We now consider the more general case where the coefficient $B$ is replaced by $B\exp(i\beta)$, which introduces a phase shift $\beta$. Analogous to Eqs. (11) and (12), the trajectory has upper and lower turning points in the $y$-direction, $y_u$ and $y_\ell$ respectively, whose loci for $0 \le \beta \le 2\pi$ are given by

$$y_u = \frac{A+B}{A-B}k_y x/k_x \quad \text{and} \quad y_\ell = \frac{A-B}{A+B}k_y x/k_x.$$

While not shown, the set of trajectories over range of phase shifts $0 \le \beta \le 2\pi$ would form caustics near the loci of $y_u$ and $y_\ell$ analogous to the caustics of Fig. 2. Note that in general the trajectory is not orthogonal to the contours of constant reduced action in Fig. 4 in contrast to Bohmian mechanics. In particular, the trajectory becomes tangent to a surface of constant reduced action at some point near the trajectory's turning point. There is a precedent for this as the trajectories for oblique reflection from a semi-infinite step potential become embedded in the contours of constant reduced action.[20]

Let us now investigate the behavior of the reduced action for $A = 1$ and $\beta = 0$ as a function of $B$. We still assume that $\hbar = 1$, $m = 1$, and $k_x = k_y = \pi/2$. By symmetry, we can reduce the scope of our examination to a square in the $x, y$-plane whose edges have one unit of length. We present selected contours of constant reduced action, $W_i$, for $\beta = 0$ and various values of $-1 \le B \le 1$ on Fig. 5 (to reduce clutter on Fig. 5 and for consistency in this paragraph with Fig. 5, we use positive and negative values of $B$ while restricting $\beta$ to zero). The straight contour in a diagonal direction of Fig. 5 for $B = 0$ is associated with the rectilinear propagation of the running wave function, $A\exp[i(k_x x + k_y y)]$ without any interference. For $A = 1$ and $B = 1$, the straight, piecewise continuous contours represent (cosine) standing-wave propagation and for $B = -1$, (sine) standing-wave propagation. By Figs. 4 and 5, we examine the behavior of the contours for reduced action as the value of $B$ changes from zero and approaches the value of $1, -1$ for $A = 1$ and $\beta = 0$. The contours of constant reduced action are initially straight for $B = 0$. When the value of $B$ becomes finite, interference exists. Then, as the finite value of $B$ increases toward the value of $A = 1$, the contours of constant reduced action become progressively more serpentine. As $B \to 1$ for $\beta = 0$, the radius of curvature goes to infinity for all points on Fig. 5 except at $(x, y) = (0, -1)$ where the radius of curvature goes to zero. An analogous process exists for $B \to -1$ on Fig. 5. When $B \to \pm A$, then the contours of constant reduced action become piecewise-continuous series of square steps associated with standing-wave propagation.

## 4 CONCLUSIONS

We conclude that the trajectory representation including Faraggi and Matone's quantum effective mass does describe interference without the need for Born's quantum probability density. In one dimension, a dichromatic particle contains all the information regarding the two interfering monochromatic wave functions. The reduced action for the dichromatic particle is a complete solution to the QSHJE and subsequently is the generator of motion for a solitary dichromatic particle which itself is a solution to the time-independent Schrödinger equation. A subsequent trajectory can be developed for the interference patterns.

This represents a simplification. First, as the dichromatic reduced action and the reduced actions for the interfering wave functions are all complete solutions to the QSHJE, each reduced action is described by the same number of nontrivial constants of integration (as $\psi_d$ is entangled, the nontrivial constants of integration for $W_1$ and $W_2$ may be redundant to some degree). Only one set of non-trivial constants of integration are needed to describe the motion of the interference patterns. Second, as the dichromatic reduced action is the generator of motion for the interference pattern, one need only derive a single trajectory with its constant coordinates to establish interference.

The trajectory representation describes the transition from running wave functions through interfering wave functions to standing wave functions. The trajectory representation shows that interference induces quantum nonlocality where pairs of interference patterns are created or annihilated. The segment of one member of the pair associated with creation or annihilation manifests retrograde motion where Faraggi and Matone's quantum effective mass is negative.

If a wave function for a free particle subject to nil potential is not monochromatic, then time reversals and the attendant pair creation and annihilations will eventually be induced. Pair creations and annihilation of interference patterns do not imply any high energy process as Faraggi and Matone's quantum effective mass for the dichromatic wave function becomes nil at these points. The trajectory representation provides a different interpretation into what Copenhagen had heretofore considered as wave-packet spreading. The trajectory representation also extends the concept of group velocity of wave packets beyond the integrity of the wave packet. It also generalizes the concept of group velocity for wave packets to cases where spectral components may have significantly different amplitudes.

The trajectory for $\psi_d(x)$ renders insight into quantum nonlocality for entangled spectral components, $\psi_1(x)$ and $\psi_2(x)$. The time reversal of the trajectories of $\psi_d(x)$ exhibit the nature of "spooky action at a distance" for entangled wave functions by inducing time reversals and retrograde motion.

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**Figure Captions**

Fig. 1. Motion, $x(t)$, of the dichromatic wave function for $\tau = 0$, $A = 1$, $B = 0.5$, $k = \pi/2$ and $\beta = 0$ as a solid line and for $\beta = \pi$ as a dashed line. The turning points are a manifestation of nonlocality in quantum mechanics. At the turning points on the left where local minima of time exist, pairs of maximum interference patterns are created. One pattern propagates with advancing motion; the other, in retrograde motion. Likewise, at the turning points on the right where local maxima of time exist, pairs of minimum interference patterns merge where one pattern has an advancing trajectory while the other has a retrograde trajectory. As an interference pattern propagates along a trajectory from a local minimum in time to a local maximum in time, the interference pattern monotonically becomes more destructive.

Fig. 2. Motion of the dichromatic wave function, $x(t)$, for $\tau = 0$, $A = 1$, $B = 0.5$, $k = \pi/2$ and $\beta = 0, \pi/4, \pi/2, \cdots, 7\pi/4$ for a set of trajectories. All trajectories are displayed as solid lines. With this spacing, the left caustic formed by the set of trajectories near their turning points of local minimum time is quite evident. The right caustic formed by the trajectories near their turning points of local maximum time is not as evident with this degree of trajectory spacing. Only advancing and never retrograde segments of the trajectories may become tangent to the caustics.

Fig. 3. Transformed effective quantum mass, $\mathcal{M}_{Q_d}$, and the transformed velocity, $\overset{\bullet}{\mathcal{X}}_d$, as function of position, $x$. While $\mathcal{M}_{Q_d}$ is continuous, $\overset{\bullet}{\mathcal{X}}_d$ is discontinuous at points where the velocity $\dot{x}$ jumps from $\pm\infty$ to $\mp\infty$. Retrograde motion is manifested by negative values of $\overset{\bullet}{\mathcal{X}}_d$.

Fig. 4. Contours of constant reduced action and trajectory on the $x, y$-plane for $y_0 = 0$, $A = 1$, $B = 0.5$, and $k_x = k_y = \pi/2$. The contours are presented as solid lines with spacing between contours of $h/4$ units of action. A trajectory launched from $(x, y) = (-2, -2)$ is presented as a dashed line.

Fig. 5. Contours of constant reduced action on the $x, y$-plane for $A = 1$, $k_x = k_y = \pi/2$, $\beta = 0$, and $B = 0, \pm 0.25, \pm 0.5, \pm 0.75, \pm 1$. The individual contours are designated by $B$. The contour of constant reduced action for $B = 1$ is piecewise continuous consisting of two straight segments overlying the edges of this Figure: one segment given by $y = 0$; the other, $x = 1$. And the contour of constant reduced action for $B = -1$ is also piecewise continuous consisting of two straight segments overlying the other edges of this Figure; one given by $x = 0$; the other, $y = -1$.